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Note

A note on the triameter of graphs

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ABSTRACT

In this article, we answer three questions from the paper (Das, 2021). We observe a tight lower bound for the triameter of trees in terms of order and number of leaves. We show that in a connected block graph any triametral triple of vertices contains a diametral pair and that any diametral pair of vertices can be extended to a triametral triple. We also present several open problems concerning the interplay between triametral triples, diametral pairs and peripheral vertices in median and distance-hereditary graphs.

1. Introduction

In [2] A. Das initiated the study of a new graph parameter

$$\mathrm{tr}(G) = \max\{d_G(a,b) + d_G(a,c) + d_G(b,c) : a, b, c \in V(G)\}$$

named the *triameter* of a connected (simple, finite) graph G. At first, a triameter was used as a parameter in [10], but explicitly named only in [8]. The main motivation for studying $\mathrm{tr}(G)$ comes from its appearance in lower bounds on radio k-chromatic number of a graph [8,11] and total domination number of a connected graph [5].

Among other results, in [2] it was shown that for any connected graph G we have $\mathrm{tr}(G) \geq g(G)$, where $g(G)$ is the girth of G. Also, the following lower bound for the triameter of a tree T with n vertices and l leaves was presented:

$$\mathrm{tr}(T) \geq \left\lceil \frac{4(n-1)}{l-1} \right\rceil . \tag{1}$$

In the final section of [2] A. Das raised four open questions concerning the triameter (which we will refer through this paper as to “Questions 1–4”):

1. The bound (1) is not tight. What is the tight lower bound for $\mathrm{tr}(T)$ for any given pair n, l?
2. Is there another lower bound for $\mathrm{tr}(G)$ for all connected graphs G in terms of parameters different from girth (it is believed that $\delta(G)$, $\Delta(G)$ will do)?
3. Is it true that any triametral triple of vertices in a tree contains a diametral pair?
4. Is it true that any diametral pair of vertices in a tree can be extended to a triametral triple?

In this article, we completely answer Question 1 by presenting a tight lower bound for $\mathrm{tr}(T)$ in terms of n, l (Theorem 1.1).

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Theorem 1.1. *Let T be a tree with $n \geq 4$ vertices and $l \geq 3$ leaves. Then*

$$\operatorname{tr}(T) \geq 6\left\lfloor \frac{n-1}{l} \right\rfloor + 2\min\{(n-1) \bmod l, 3\}.$$

Moreover, this bound is tight for any given pair n, l.

We also give affirmative answers to Questions 3 and 4 not only for trees, but for all connected block graphs (Theorem 1.2).

Theorem 1.2. *Let G be a connected block graph, $d = d_G$ and $a, b, c, x, y \in V(G)$ such that $d(a, b, c) = \operatorname{tr}(G)$, $d(x, y) = \operatorname{diam}(G)$. Then*

$$\max\{d(a, b), d(a, c), d(b, c)\} = \operatorname{diam}(G) \text{ and}$$
$$\max\{d(a, x, y), d(b, x, y), d(c, x, y)\} = \operatorname{tr}(G).$$

Question 2 still remains unanswered and can be studied further. In the last section of the paper, we formulate several questions concerning the interplay between the triametral triples, diametral pairs and peripheral vertices in median and distance-hereditary graphs.

2. Preliminaries

All graphs under consideration are simple and finite. A graph is *connected* if any pair of its vertices can be joined by a path. The vertex set of a connected graph G is equipped with the "shortest-path" metric d_G, where $d_G(u, v)$ is the length of a shortest $u - v$ path in G. The *diameter* of a connected graph G is the value $\operatorname{diam}(G) = \max\{d_G(u, v) : u, v \in V(G)\}$. A pair of vertices $u, v \in V(G)$ in a connected graph G is called *diametral* if $d_G(u, v) = \operatorname{diam}(G)$. A vertex $u \in V(G)$ is called *peripheral* if it belongs to some diametral pair.

For a triple of vertices $u, v, w \in V(G)$ in a connected graph G we have

$$d_G(u, v, w) = d_G(u, v) + d_G(u, w) + d_G(v, w).$$

The *triameter* of a connected graph G is defined as the value

$$\operatorname{tr}(G) = \max\{d_G(u, v, w) : u, v, w, \in V(G)\}.$$

The three vertices $u, v, w \in V(G)$ is *triametral* if $d_G(u, v, w) = \operatorname{tr}(G)$.

Let $u, v \in V(G)$ be a pair of vertices in a connected graph G. Now

$$[u, v]_G = \{x \in V(G) : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$$

for the *metric interval* between u and v. A connected graph is called *median* [9] if $|[u, v]_G \cap [u, w]_G \cap [v, w]_G| = 1$ for any triple of vertices $u, v, w \in V(G)$.

A *tree* is a connected graph without cycles. Note that any tree is a median graph. A vertex of degree one in a graph is called its *leaf*. The set of all leaves in a tree T is denoted by $L(T)$. If u is a leaf in a tree, then the unique vertex v adjacent to u is called its *support vertex*.

A *connected component* of a graph is its maximal connected subgraph. A vertex whose deletion increases the number of connected components in a graph is called its *cut-vertex*. A graph is *biconnected* if it has no cut-vertices. A *block* of a graph is its maximal biconnected subgraph. A *block graph* $B(G)$ of a given graph G is the intersection graph of the collection of all blocks in G. A graph H is called a *block graph* if it is isomorphic to $B(G)$ for some G. It is well-known fact that a graph is a block graph if and only if each of its block is complete [4]. As a corollary, we obtain that every tree is a block graph.

The following bounds for the triameter of a graph can be easily derived from the definition.

Proposition 2.1. *For any connected graph G we have*

$$2\operatorname{diam}(G) \leq \operatorname{tr}(G) \leq 3\operatorname{diam}(G).$$

A connected graph G is called *symmetric even* if for each vertex $u \in V(G)$ there exists a unique vertex $u' \in V(G)$ with $[u, u']_G = V(G)$ (see [3]). Note that for any u such a vertex u' is always unique. The vertex u' is called *symmetric vertex* for u. For a symmetric even graph G, $d_G(u, u') = \operatorname{diam}(G)$ for any vertex $u \in V(G)$. We have the following result for the triameter of symmetric even graphs.

Proposition 2.2. *For any symmetric even graph G $\operatorname{tr}(G) = 2\operatorname{diam}(G)$.*

Proof. Let $x, y, z \in V(G)$ be a triametral triple of vertices in G. Then $2\operatorname{diam}(G) \leq \operatorname{tr}(G) = d_G(x, y, z) = d_G(x, y) + d_G(x, z) + d_G(y, z) = d_G(x, x') - d_G(x', y) + d_G(x, x') - d_G(x', z) + d_G(y, z) = 2d_G(x, x') - d_G(x', y) - d_G(x', z) + d_G(y, z) \leq 2d_G(x, x') = 2\operatorname{diam}(G)$, where x' is the symmetric vertex for x. $\square$

Let $n \in \mathbb{N}$. The *n-cube* is a graph Q_n with $V(Q_n) = \{0, 1\}^n$ and $E(Q_n) = \{xy :$ there is a unique $i,\ 1 \le i \le n$ with $x_i \ne y_i\}$. Note that every n-cube is a median as well is a symmetric even graph.

Corollary 2.3. *For any $n \in \mathbb{N}$ we have $\mathrm{tr}(Q_n) = 2n$.*

Proof. Since $\mathrm{diam}(Q_n) = n$ and Q_n is symmetric even, the equality $\mathrm{tr}(Q_n) = 2n$ immediately follows from Proposition 2.2. □

Note that Corollary 2.3 also can be deduced from the following observation about the triameter of Cartesian product of two connected graphs.

Proposition 2.4 (*[8]*). *For any two connected graphs G and H, $\mathrm{tr}(G\Box H) = \mathrm{tr}(G) + \mathrm{tr}(H)$.*

Now, since Q_n is a Cartesian product of n copies of K_2, the desired equality $\mathrm{tr}(Q_n) = 2n$ easily follows.

Corollary 2.5. *Every pair of vertices in a symmetric even graph can be extended to a triametral triple.*

Proof. If $u, v \in V(G)$ is a pair of vertices in a symmetric even graph G, then Proposition 2.2 asserts $d_G(u, v, v') = d_G(u, v) + d_G(u, v') + d_G(v, v') \ge 2d_G(v, v') = 2\,\mathrm{diam}(G) = \mathrm{tr}(G)$, where v' is the symmetric vertex for v. □

In particular, Question 4 is valid for symmetric even graphs. However, Question 3 does not seem precise for symmetric even graphs as the 3-cube Q_3 has a triametral triple of vertices $x, y, z \in V(Q_3)$ with $d_{Q_3}(x, y) = d_{Q_3}(x, z) = d_{Q_3}(y, z) = 2 < 3 = \mathrm{diam}(Q_3)$.

3. Main results

3.1. An optimal lower bound for the triameter of trees

In this subsection we prove Theorem 1.1.

Proof of Theorem 1.1. We use induction on $n \ge 4$. If $n = 4$, then $T = K_{1,3}$, $l = 3$ and $\mathrm{tr}(T) = 6 = 6\lfloor\frac{n-1}{l}\rfloor + 2\min\{(n-1) \bmod l, 3\}$. Now suppose that $n \ge 5$. Consider the tree $T' = T\backslash L(T)$ and put $l' = |L(T')|$. Note that $l' \le l$. If $l' = 1$, then $T = K_{1,n-1}$, $l = n-1$ and $\mathrm{tr}(T) = 6 = 6\lfloor\frac{n-1}{l}\rfloor + 2\min\{(n-1) \bmod l, 3\}$. If $l' = 2$, then T is bistar and $l = n-2$. Since $n \ge 5$, $T \ne P_4$ and therefore $\mathrm{tr}(T) = 8 = 6\lfloor\frac{n-1}{l}\rfloor + 2\min\{(n-1) \bmod l, 3\}$. Thus, assume $l' \ge 3$. By induction assumption, $\mathrm{tr}(T') \ge 6\lfloor\frac{n-l-1}{l'}\rfloor + 2\min\{(n-l-1) \bmod l', 3\}$.

Claim: $\mathrm{tr}(T) = \mathrm{tr}(T') + 6$.

Let $d_{T'}(x, y, z) = \mathrm{tr}(T')$ for some $x, y, z \in V(T')$. Then $x, y, z \in L(T')$ and hence there are three vertices $x', y', z' \in L(T)$ with $xx', yy', zz' \in E(T)$. We have $\mathrm{tr}(T) \ge d_T(x', y', z') = d_{T'}(x, y, z) + 6 = \mathrm{tr}(T') + 6$. Similarly, if $d_T(a, b, c) = \mathrm{tr}(T)$, then $a, b, c \in L(T)$. Let a', b', c' be the corresponding support vertices in T for the leaves a, b, c, respectively. Then $\mathrm{tr}(T) = d_T(a, b, c) = d_{T'}(a', b', c') + 6 \le \mathrm{tr}(T') + 6$. This proves the claim.

Put $m = (n-1) \bmod l$ and $m' = (n-l-1) \bmod l'$ for the sake of simplicity. Note that $n-l-1 \ge m$. By our Claim,

$$\begin{aligned}
&\mathrm{tr}(T) - (6\lfloor\frac{n-1}{l}\rfloor + 2\min\{(n-1) \bmod l, 3\})\\
&= \mathrm{tr}(T') + 6 - 6\lfloor\frac{n-1}{l}\rfloor - 2\min\{m, 3\}\\
&\ge 6\lfloor\frac{n-l-1}{l'}\rfloor + 2\min\{(n-l-1) \bmod l', 3\} + 6 - 6\lfloor\frac{n-1}{l}\rfloor - 2\min\{m, 3\}\\
&= 6\lfloor\frac{n-l-1}{l'}\rfloor + 2\min\{m', 3\} + 6 - 6\lfloor\frac{n-1}{l}\rfloor - 2\min\{m, 3\}\\
&= 6(\lfloor\frac{n-l-1}{l'}\rfloor - \lfloor\frac{n-1}{l}\rfloor + 1) + 2(\min\{m', 3\} - \min\{m, 3\})\\
&= 6(\lfloor\frac{n-l-1}{l'}\rfloor - \lfloor\frac{n-l-1}{l}\rfloor) + 2(\min\{m', 3\} - \min\{m, 3\}).
\end{aligned}$$

Since $l' \le l$, $\lfloor\frac{n-l-1}{l'}\rfloor \ge \lfloor\frac{n-l-1}{l}\rfloor$. If $\lfloor\frac{n-l-1}{l'}\rfloor = \lfloor\frac{n-l-1}{l}\rfloor$, then $\frac{n-l-1-m'}{l'} = \frac{n-l-1-m}{l}$. Therefore, $m' = \frac{(n-l-1)(l-l')+ml'}{l} \ge \frac{m(l-l')+ml'}{l} = m$. Hence, in case $\lfloor\frac{n-l-1}{l'}\rfloor = \lfloor\frac{n-l-1}{l}\rfloor$, we have $2(\min\{m', 3\} - \min\{m, 3\}) \ge 0$ and thus $\mathrm{tr}(T) - (6\lfloor\frac{n-1}{l}\rfloor + 2\min\{(n-1) \bmod l, 3\}) \ge 0$. Finally, if $\lfloor\frac{n-l-1}{l'}\rfloor > \lfloor\frac{n-l-1}{l}\rfloor$, then

$$\begin{aligned}
&\mathrm{tr}(T) - (6\lfloor\frac{n-1}{l}\rfloor + 2\min\{(n-1) \bmod l, 3\})\\
&\ge 6(\lfloor\frac{n-l-1}{l'}\rfloor - \lfloor\frac{n-l-1}{l}\rfloor) + 2(\min\{m', 3\} - \min\{m, 3\})\\
&> 6 + 2(\min\{m', 3\} - \min\{m, 3\}) \ge 6 - 2\cdot 3 = 0
\end{aligned}$$

as well. In all the cases, $\mathrm{tr}(T) \ge 6\lfloor\frac{n-1}{l}\rfloor + 2\min\{(n-1) \bmod l, 3\}$ which proves the induction step.

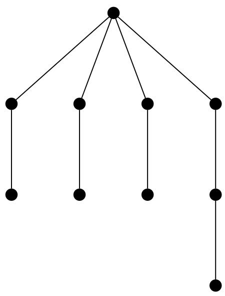

Fig. 1. Example of a tree $T_{n,l}$.

Now we prove that the obtained bound is tight. To do this, fix $n \geq 4$ and $3 \leq l \leq n-1$. Construct a tree $T_{n,l}$ as follows: start with a star $K_{1,l}$, then fix a set of edges $E' \subset E(K_{1,l})$ with $|E'| = m = (n-1) \bmod l$ and subdivide each of them by $\lfloor \frac{n-l-1}{l} \rfloor + 1$ new vertices; each other edge in $K_{1,l}$ is subdivided by $\lfloor \frac{n-l-1}{l} \rfloor$ new vertices to obtain $T_{n,l}$. By construction, $T_{n,l}$ has

$$\begin{aligned} &1 + l + \lfloor \frac{n-l-1}{l} \rfloor \cdot l + (n-1) \bmod l \\ &= 1 + l + \lfloor \frac{n-l-1}{l} \rfloor \cdot l + (n-l-1) \bmod l \\ &= 1 + l + n - l - 1 = n \end{aligned}$$

vertices and l leaves.

We must consider four cases depending on the remainder m. If $m \geq 3$, then fix three vertices a, b, c each incident to some edge from E'. It holds $\mathrm{tr}(T_{n,l}) = d_{T_{n,l}}(a, b, c) = 3 \cdot 2 \cdot (\lfloor \frac{n-l-1}{l} \rfloor + 2) = 6\lfloor \frac{n-1}{l} \rfloor + 6 = 6\lfloor \frac{n-1}{l} \rfloor + 2\min\{m, 3\}$. If $m = 2$, then fix two leaf vertices a, b each incident to some edge from E' and another leaf vertex c from $T_{n,l}$. In this case,

$$\begin{aligned} &\mathrm{tr}(T_{n,l}) = d_{T_{n,l}}(a, b, c) = d_{T_{n,l}}(a, b) + d_{T_{n,l}}(a, c) + d_{T_{n,l}}(b, c) \\ &= 2 \cdot (\lfloor \frac{n-l-1}{l} \rfloor + 2) + 2 \cdot (2\lfloor \frac{n-l-1}{l} \rfloor + 3) \\ &= 6\lfloor \frac{n-1}{l} \rfloor + 4 = 6\lfloor \frac{n-1}{l} \rfloor + 2\min\{m, 3\}. \end{aligned}$$

Further, if $m = 1$, then fix a leaf vertex a which is incident to the unique edge from E' and two other leaf vertices b, c from $T_{n,l}$. We have

$$\begin{aligned} &\mathrm{tr}(T_{n,l}) = d_{T_{n,l}}(a, b, c) = d_{T_{n,l}}(a, b) + d_{T_{n,l}}(a, c) + d_{T_{n,l}}(b, c) \\ &= 2 \cdot \lfloor \frac{n-l-1}{l} \rfloor + 3 + 2 \cdot 2 \cdot (\lfloor \frac{n-l-1}{l} \rfloor + 1) \\ &= 6\lfloor \frac{n-1}{l} \rfloor + 2 = 6\lfloor \frac{n-1}{l} \rfloor + 2\min\{m, 3\}. \end{aligned}$$

Finally, for $m = 0$, then for any triple a, b, c of leaf vertices from $T_{n,l}$ it holds $\mathrm{tr}(T_{n,l}) = 3 \cdot 2 \cdot (\lfloor \frac{n-l-1}{l} \rfloor + 1) = 6\lfloor \frac{n-1}{l} \rfloor = 6\lfloor \frac{n-1}{l} \rfloor + 2\min\{m, 3\}$. In all the four cases the desired equality holds. □

Fig. 1 contains a tree $T_{n,l}$ from the proof of Theorem 1.1 for $n = 10$, $l = 4$.

3.2. From triameter to diameter and vice versa in block graphs

Consider the graph G represented by Fig. 2. It is easy to see that $\mathrm{tr}(G) = d_G(a, b, c) = 12$ and $\mathrm{diam}(G) = d_G(x, y) = 5$. Also, $d_G(a, b) = d_G(a, c) = d_G(b, c) = 4$ and $d_G(x, y, z) = 10$ for all $z \in V(G)$ (since $[x, y]_G = V(G)$). In other words, the triametral triple a, b, c does not contain a diametral pair in G as well as diametral pair x, y cannot be extended to a triametral triple in G.

To show that any triametral triple of vertices in a block graph contains a diametral pair and that any diametral pair of vertices can be extended to a triametral triple, we will use the next metric characterization of block graphs.

Theorem 3.1 ([7]). *A connected graph G is a block graph if and only if its metric d_G satisfies the "4-point condition": for any $x, y, z, t \in V(G)$ it holds*

$$d_G(x, y) + d_G(z, t) \leq \max\{d_G(x, z) + d_G(y, t), d_G(x, t) + d_G(y, z)\}.$$

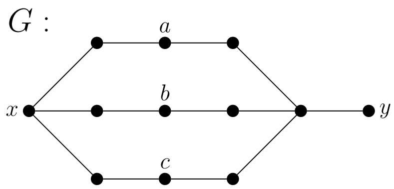

Fig. 2. The triametral triple a, b, c does not contain a diametral pair and the diametral pair x, y cannot be extended to a triametral triple.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Since G is a connected block graph, from Theorem 3.1 it follows that $d(x, y) + d(a, b) \leq \max\{d(x, a) + d(y, b), d(x, b) + d(y, a)\}$. Without loss of generality, we can assume that

$$d(x, y) + d(a, b) \leq d(x, a) + d(y, b). \tag{2}$$

If $d(a, y) \geq d(y, b)$, then

$$\begin{aligned}0 &\geq d(a, x, y) - d(a, b, c) = d(a, x) + d(a, y) + d(x, y) - d(a, b) - d(a, c) - d(b, c)\\ &= (d(a, x) - d(a, b)) + d(a, y) + d(x, y) - d(a, c) - d(b, c)\\ &\geq (d(x, y) - d(y, b)) + d(a, y) + d(x, y) - d(a, c) - d(b, c)\\ &= (d(a, y) - d(y, b)) + 2d(x, y) - d(a, c) - d(b, c)\\ &\geq 2\,\mathrm{diam}(G) - d(a, c) - d(b, c) \geq 0.\end{aligned}$$

Hence, $d(a, x, y) = d(a, b, c) = \mathrm{tr}(G)$ and $d(a, c) = d(b, c) = \mathrm{diam}(G)$.

If $d(b, x) \geq d(x, a)$, then

$$\begin{aligned}0 &\geq d(b, x, y) - d(a, b, c) = d(b, x) + d(b, y) + d(x, y) - d(a, b) - d(a, c) - d(b, c)\\ &(d(b, y) - d(a, b)) + d(b, x) + d(x, y) - d(a, c) - d(b, c)\\ &\geq (d(x, y) - d(x, a)) + d(b, x) + d(x, y) - d(a, c) - d(b, c)\\ &= (d(b, x) - d(x, a)) + 2d(x, y) - d(a, c) - d(b, c)\\ &\geq 2\,\mathrm{diam}(G) - d(a, c) - d(b, c) \geq 0.\end{aligned}$$

Thus, in this case also $d(b, x, y) = d(a, b, c) = \mathrm{tr}(G)$ and $d(a, c) = d(b, c) = \mathrm{diam}(G)$.

Now suppose $d(a, y) < d(y, b)$ and $d(b, x) < d(x, a)$. Then $d(a, y) + d(b, x) < d(y, b) + d(x, a)$ implying that $d(y, b) + d(x, a) \leq d(x, y) + d(a, b)$ (again, see Theorem 3.1). Combining this inequality with (2), we obtain the equality

$$d(x, y) + d(a, b) = d(x, a) + d(y, b). \tag{3}$$

In a similar manner, we can consider two sums $d(x, y) + d(a, c)$, $d(x, y) + d(a, c)$ and apply Theorem 3.1 to each of them. Hence, we restrict ourselves to the case where the following equalities hold:

$$d(x, y) + d(a, c) = d(a, x) + d(c, y) \text{ or} \tag{4}$$

$$d(x, y) + d(a, c) = d(a, y) + d(c, x) \tag{5}$$

and

$$d(x, y) + d(b, c) = d(b, x) + d(c, y) \text{ or} \tag{6}$$

$$d(x, y) + d(b, c) = d(b, y) + d(c, x). \tag{7}$$

If (5) holds, then using (3), we obtain

$$\begin{aligned}0 &\geq d(a, x, y) - d(a, b, c) = d(a, x) + d(a, y) + d(x, y) - d(a, b) - d(a, c) - d(b, c)\\ &= (d(x, y) + d(a, b) - d(y, b)) + (d(x, y) + d(a, c) - d(c, x))\\ &+ d(x, y) - d(a, b) - d(a, c) - d(b, c)\\ &= 3\,\mathrm{diam}(G) - d(y, b) - d(c, x) - d(b, c) \geq 0.\end{aligned}$$

Therefore, $d(a, x, y) = \mathrm{tr}(G)$ and $d(b, c) = \mathrm{diam}(G)$. If (6) holds, then using (3), we similarly obtain $d(b, x, y) = \mathrm{tr}(G)$ and $d(a, c) = \mathrm{diam}(G)$.

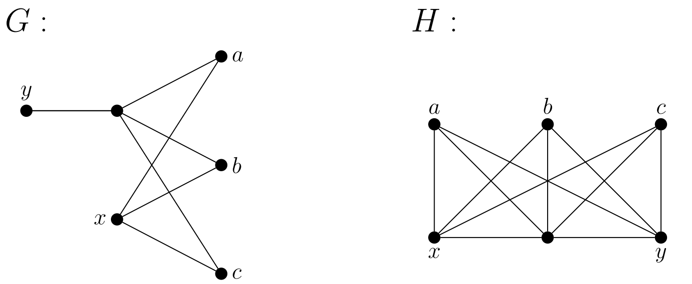

Fig. 3. The triametral triple a, b, c in G does not contain a peripheral vertex and the peripheral vertex x in H cannot be extended to a triametral triple.

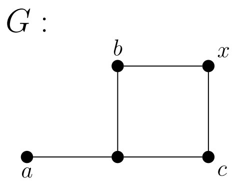

Fig. 4. The triametral triple a, b, c in a median graph G does not contain a diametral pair.

Finally, assume that (4) and (7) hold. Then

$$\begin{aligned}0 &\geq d(c,x,y) - d(a,b,c) = d(c,x) + d(c,y) + d(x,y) - d(a,b) - d(a,c) - d(b,c)\\ &= (d(x,y) + d(b,c) - d(b,y)) + (d(x,y) + d(a,c) - d(a,x))\\ &+ d(x,y) - d(a,b) - d(a,c) - d(b,c)\\ &= 3\,\mathrm{diam}(G) - d(b,y) - d(a,x) - d(a,b) \geq 0.\end{aligned}$$

Hence, in this case $d(c,x,y) = \mathrm{tr}(G)$ and $d(a,b) = \mathrm{diam}(G)$. □

One natural generalization of block graphs are *distance-hereditary* graphs. These are connected graphs in which every induced connected subgraph is isometric [6]. As follows from [1], a connected graph G is distance hereditary if and only if for any four its vertices $a, b, c, d \in V(G)$ the two sums namely $d_G(a,b) + d_G(c,d)$, $d_G(a,c) + d_G(b,d)$, $d_G(a,d) + d_G(b,c)$ are equal. Combining this characterization with Theorem 3.1, we obtain that every block graph is distance-hereditary. However, the statement of Theorem 1.2 cannot be extended to distance-hereditary graphs. To see this, consider the two graphs G and H in Fig. 3. Indeed, G and H are both distance-hereditary, but the triametral triple a, b, c in G does not contain even a peripheral vertex (since $\mathrm{diam}(G) = d_G(x,y) = 3$). Similarly, the peripheral vertex x (and thus a diametral pair x, y) in H cannot be extended to a triametral triple (since $\mathrm{tr}(H) = d_G(a,b,c) = 6$).

4. Open questions

Let $\mathcal{P}$ be a family of graphs. We list here several possible generalizations of Questions 3,4 which should be worth investigating for various classes of graphs $\mathcal{P}$.

Question 3': Is it true that any triametral triple of vertices of a graph G from the family $\mathcal{P}$ contains a peripheral vertex?

Question 4': Is it true that any peripheral vertex of a graph G from the family $\mathcal{P}$ can be extended to a triametral triple?

As can be seen from the graph in Fig. 4, Question 3 does not hold for median graphs: $\mathrm{tr}(G) = d_G(a,b,c) = 6$, but the triple a, b, c does not contain diametral pairs as $\mathrm{diam}(G) = d_G(a,x) = 3$. However, the triple a, b, c contains a peripheral vertex a. Therefore, we formulate the next problem:

1. Does Question 3' hold for median graphs?

Also, we do not know whether every triametral triple of vertices in a median graph contains a diametral pair, thus formulating our second problem:

2. Does Question 4 hold for median graphs?

It is worth noting that Questions 3' and 4' do not hold for modular graphs (these are connected graphs G in which $[u,v]_G \cap [u,w]_G \cap [v,w]_G \neq \emptyset$ for any three vertices $u, v, w \in V(G)$), which are the natural generalization of median graphs. Indeed, the graph G in Fig. 3 is modular, however as it was already mentioned, the triametral triple a, b, c does not contain a peripheral vertex. Also, the modular graph $K_{2,3}$ contains a peripheral vertex which does not belong to a triametral triple.

Finally, Questions 3' and 4' also do not hold for distance-hereditary graphs (again, see the graphs in Fig. 3). At the end of the paper we propose our final problem, the "alternative" for distance-hereditary graphs:

3. Is it true that for a distance-hereditary graph at least one of Questions 3' or 4 hold?

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